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PRIMARY SCHOOL STUDENTS' STRUCTURE AND LEVELS OF ABILITIES IN TRANSFORMATIONAL GEOMETRY

ABSTRACT

The aim of this study was twofold: 1) to investigate the components that synthesize primary school students' transformational geometry ability, and 2) to describe levels of abilities in transformational geometry. A test aiming to measure abilities in geometric transformations (translations, reflections and rotations) was administered to 166 primary school students of the fourth, fifth and sixth grade. Confirmatory factor analyses indicated the existence of four distinct factors of ability for each of the three geometric transformations, suggesting that they have similar structure. Furthermore, RASCH measurement methodology was used to create a hierarchy of these tasks. The scale was interpreted as a progression of five levels of abilities in transformational geometry: wholistic image, motion of an object, mapping of an object, mapping of the plane, and self-regulated mapping of the plane. Students' development of understanding is interpreted in light of the theoretical framework of geometrical paradigms and geometric work space (GWS).

KEY WORDS

transformational geometry, geometric work space, visualization, dimensional deconstruction

1. Introduction

The growing emphasis on geometry teaching during the last few decades has modified its traditionally Euclidian-based content, by introducing new types of geometry, such as transformational geometry (Jones, 2002). Transformational geometry refers to mental or physical transformation of shapes. The most common types of geometric transformations in literature and in primary school textbooks are translation, axial reflection and point rotation. According to NCTM 's *Principals and Standards for School Mathematics* (2000), "Instructional programs from kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (p.41).

During the last decade, there is growing emphasis on transformational geometry in the field of mathematics education (Hollebrands, 2003; Portnoy, Grundmeier & Graham, 2006; Yanik & Flores, 2009). Some of the main reasons are that: 1) it is considered important in supporting children's development of geometric and spatial thinking (Hollebrands, 2003), and 2) it is related to a variety of activities in academic and every-day life, such as geometrical constructions, art, architecture, carpentry, electronics, mechanics, geography and navigation (Boulter & Kirby, 1994).

Research in transformational geometry during the last few years has focused in studying the development of knowledge and understanding of transformations (Molina, 1990; Soon, 1989; Thaqi, Giménez, & Rosich, 2011; Yanik & Flores, 2009) and various theoretical frameworks have been used (Hollebrands, 2003; Molina, 1990; Portnoy et al., 2006; Soon, 1989). However, the components that synthesize this ability appear not to have been clearly defined in literature yet. This seems to be critical in order to study the development of knowledge and understanding of transformations. According to Kidder (1976), performing geometric transformations is a multi-faceted mental operation.

Based on the pilot results of a large scale project investigating students' ability in geometric transformations (translation, reflection, rotation), this paper aspires to investigate the structure and development of primary school students' ability in these concepts. Hence, its aim is twofold: 1) to investigate the components that synthesize primary school students' transformational geometry ability, drawing on the findings of previous research studies, and 2) to describe primary school students' levels of abilities in transformational geometry, drawing on theoretical frameworks for understanding geometry and interpreting figures. Specifically, we draw on the notions of geometrical paradigms and GWS (Kuzniak, 2006), and dimensional deconstruction (Duval, 2005) to interpret primary school students' development of knowledge and understanding of geometric transformations.

2. Literature Review

2.1. *The development of knowledge and understanding of transformational geometry concepts*

One of the first debates in the transformational geometry field of research was the development of learning geometric transformations. While Moyer (1978) suggests that translation is at least as easy as reflection, even though reflection can be considered mathematically primitive, Schultz and Austin (1983) suggest that translations seem to be the easiest transformations for students and that the direction of the transformation influences the relative difficulty of rotations or reflections. However, these studies used only one type of task, that of performing a transformation.

The first attempts that used a variety of tasks to study the development of transformational geometry concepts as a sequence of levels were based on the van Hiele model of geometric understanding (Molina, 1990; Soon, 1989). These studies included a variety of tasks, matched to each level, such as performing transformations, visual recognition of transformations, understanding and relating the properties of transformations. However, the components that synthesize this ability have not been confirmed in literature. Moreover, these studies only focused on the type of task, ignoring previous findings regarding the order of difficulty in transformations.

Edwards (2003) proposed another model, which discriminates between two qualitatively different conceptions of geometric transformations: 1) motion, where the plane is conceived as a background and geometric figures are manipulated on top of it, and 2) mapping, where a transformation can be considered as a special function that maps all points in the plane to other points while preserving some properties and changing others.

2.2. *The geometric work space and the geometrical paradigms*

The GWS (Kuzniak, 2006) is the place organized to ensure the geometrical work. From an epistemological view, it puts the three following components in a network: a) the real and concrete objects that serve as material support, b) the artefacts like drawing tools and software, and c) a theoretical system of reference. Following Duval (1995), the cognitive processes for using these components in geometrical problem solving are: a) a visualization process with regard to space representation and material support, b) a construction process determined by the instruments and geometrical configurations, and c) a discursive reasoning process that conveys argumentation and proof. This study will emphasize on the visualization process, and specifically to the idea of dimensional deconstruction introduced by Duval (2005), which is described further on.

In school geometry, there are three GWS levels to describe geometric work: reference, appropriate, and personal GWS (Kuzniak, 2012). This paper will focus on the personal GWS. When a problem is posed to an actual individual (pupil, student or teacher), the problem is handled within his/her personal GWS, which generally depends on the cognitive abilities (i.e. knowledge) of the person. Houdement and Kuzniak (1999) describe the way in which three different paradigms could explain the different forms of geometry. A paradigm is composed of a theory that guides observation, activity and judgment, and permits new knowledge production (Houdement, 2007). Two of these, Geometry I and Geometry II, are important for geometric work up to secondary school. *Geometry I* finds its validation in the material and tangible world. *Geometry II* is built on a model that approaches reality without being fused with it (Kuzniak & Rauscher, 2011). Both Geometries have a close link to the real world, but in different ways (Kuzniak, 2012).

2.3. *Visualization and dimensional deconstruction*

There are two ways of looking at figures and recognizing what they stand for: the natural and the mathematical (Duval, 2011). One important issue in the learning of geometry in primary and secondary school is to identify the figural units which can be discriminated in any constructed figure. According to Duval (2011), visualization ability in geometry is closely related to the ability of recognizing all figural units that can be mathematically relevant.

Dimensional deconstruction describes the transition of a drawing seen as a tangible object to the figure conceived as a generic and abstract object (Duval, 2005). For example, a figure can be seen as a 2D-object (a triangle as an area), a set of 1D-objects (sides) or 0D-objects (vertices). While in the natural way perception focuses exclusively on 1D, 2D or 3D/3D figural units, just like material object, the mathematical way requires the dimensional deconstruction of any shape into figural units of 1D or 0D/2D. According to Duval (2011) becoming aware of the different ways of looking at figures is prior to the knowledge of the classical basic figures.

3. METHODOLOGY

3.1. *Participants*

The participants of the study were 166 primary school students (78 boys and 88 girls). In order to be able to study a wider spectrum of primary school students' development of transformational geometry abilities, the students that were selected for this study came from three successive grades of primary school. Specifically, fifty-two were fourth-graders, fifty-three were fifth-graders and sixty-one were sixth graders.

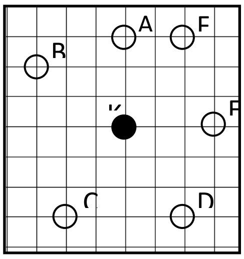
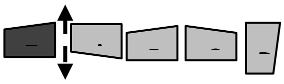
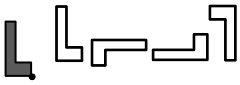
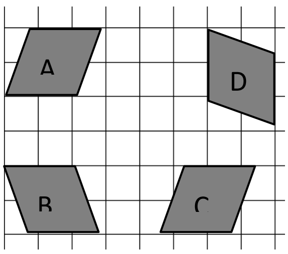
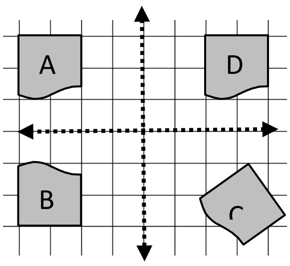
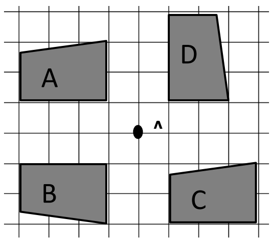
3.2. *Instrument and procedure*

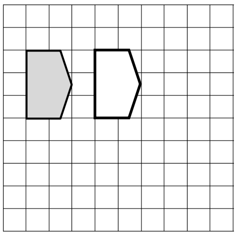
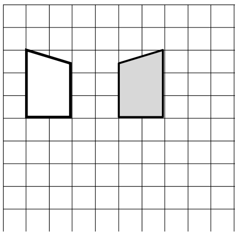
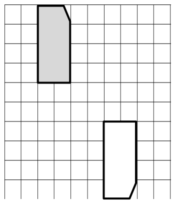
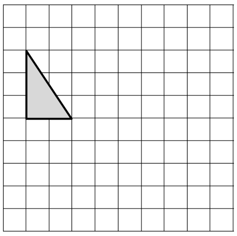
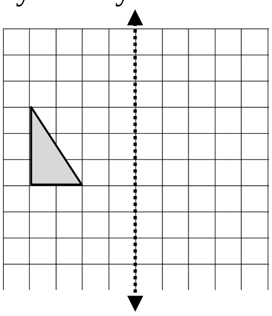
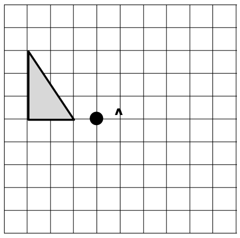
The instrument used in the study was a transformational geometry test, developed especially for the purpose of the project. The test had three sections: the first was about translation, the second about reflection and the third about rotation. Each section included four different types of tasks (see Table 1 for examples): 1) recognising the image of a translation/reflection/rotation among other choices, 2) recognising a translation/reflection/rotation among other choices, 3) identifying the parameters of a given translation/reflection/rotation, and 4) constructing the image for a given translation/reflection/rotation. For each type, at least three tasks were given: one in horizontal, one in vertical and one in diagonal direction. In types 3 and 4, there was an additional task with overlapping image and in type 4 an additional task with an unfamiliar shape in horizontal direction. The tasks were split and administered to all students in two equally difficult parts. Students were given forty minutes to complete each part of the test. To avoid practice effects, half of the students received one part of the test first, while the rest of the students received the other part. Since instruction in geometric transformations in Cyprus is not emphasized in the curriculum and mainly focuses on the concept of reflection through symmetry, operational definitions and examples were given to the students before completing the tests.

After completing the test, students' responses were graded. Types of tasks 1) recognising the image of a transformation (three tasks for each transformation), and 2) recognising a transformation (four tasks for each transformation) were multiple choice, and were graded with 1 for correct and 0 for incorrect responses. In type 3, identifying the parameters for a given translation, and type 4 tasks, constructing the image for a transformation, 0 marks were given for incorrect response and 1 for correct. Partial credit was given to responses with some correct elements. Items with no response were coded as 0.

TABLE 1

Examples of tasks in the transformational geometry ability test

Type of task	Translation example	Reflection example	Rotation example
Recognition of a transformation image	Which of the following images is the translation of the pre-image K, when it translates 3 units up? A C D E 	Which of the following shapes is the reflection of shape Z over a vertical line of symmetry? 	Which of the following shapes is the rotation of the grey figure at ¼ of a turn? 
Recognition of a transformation	Which of the following pairs of shapes show a translation? a) A and D b) B and C c) C and D d) A and C 	Which of the following pairs of shapes show a reflection? a) A and D b) B and C c) B and A d) C and D 	Which of the following pairs of shapes show a rotation? a) A and D b) B and C c) C and D d) A and C 

Defining of a transformation's parameters	Give the instructions for the translation of the shaded figure to the position of the white figure. 	Draw the line of symmetry for every case. 	Find the point of rotation and the fraction that shows how much the shape turned to the right.  1/4 2/4 3/4
Construction of an image under transformation	Translate 4 units to the right. 	Draw the reflection of each shape over the given line of symmetry. 	Rotate the shape ¼ of a turn to the right. 

3.3. *Statistical procedures*

For testing the fit of the theoretical model regarding the structure of transformational geometry ability, the MPLUS software was used with Maximum Likelihood (ML) estimator. More than one fit indices were used to evaluate the extent to which the data fit the theoretical model under investigation. More specific, the fit indices and their optimal values were: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, since a significant chi-square indicates lack of satisfactory model fit, (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2004).

For investigating the development of knowledge and understanding of transformational geometry concepts, the RASCH measurement model was used. This methodology is based on the assumption that the difference between item difficulty and person ability should govern the probability of any person being successful on any particular item, and ranks both the persons and the items on the same scale, based on these probabilities. The model fit statistics are two: 1) the infit (weighted) mean square statistic, and 2) the outfit (unweighted) mean square statistic. The normalized statistics (using the Wilson–Hilferty transformation), infit t and outfit t, have a mean near zero and a standard deviation near one when the data conform to the measurement model. No items or persons should have a zero score or a perfect score.

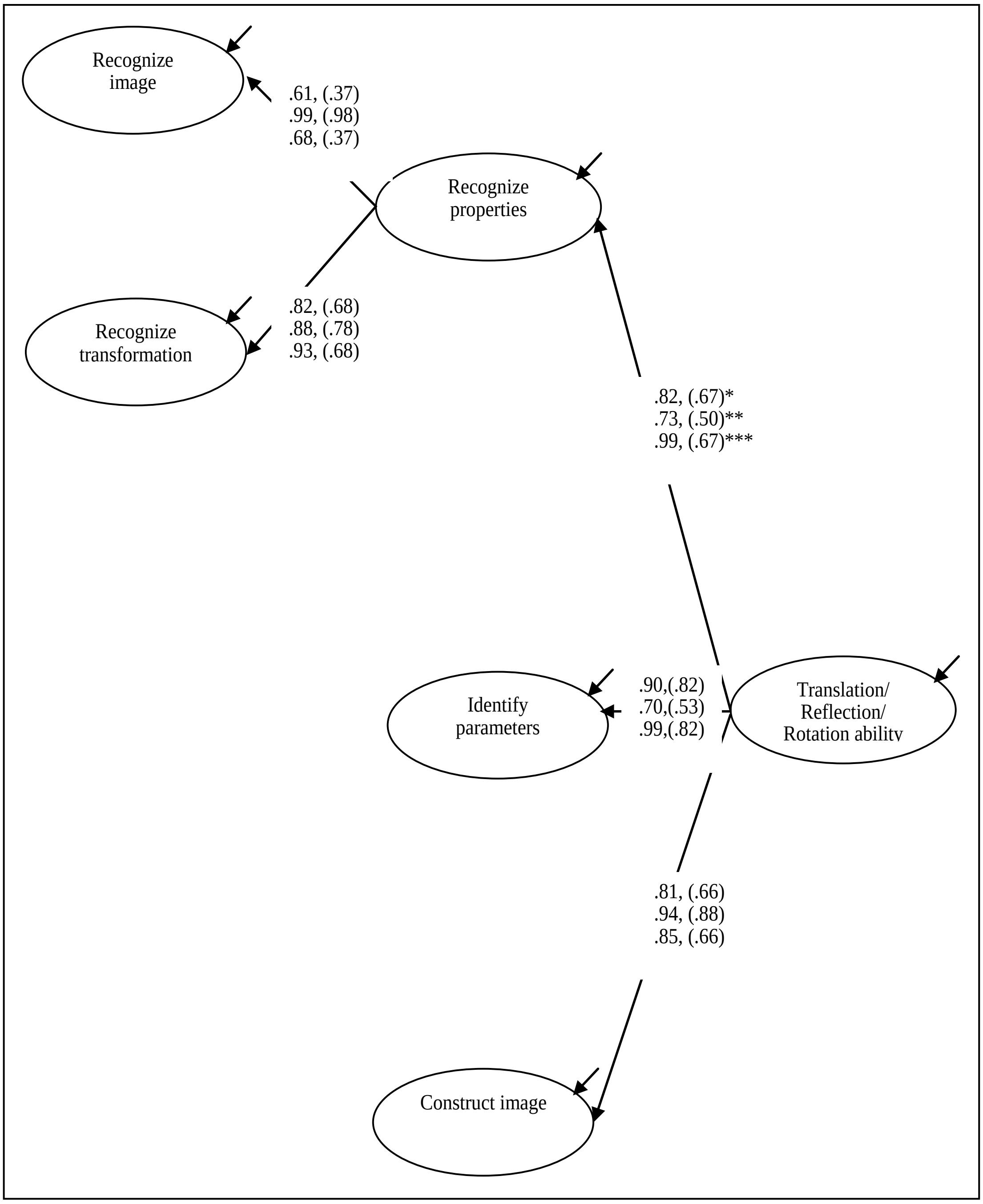
For this study, the dichotomous model of RASCH analysis was used, which predicts the conditional probability of a binary outcome (correct/incorrect), given the person's ability and the item's difficulty. Therefore, for this analysis, the data were recoded into 1 mark for correct responses and 0 for incorrect or partially correct responses.

4. RESULTS

The first aim of this study was to investigate the components that synthesize primary school students' transformational geometry ability, and the structure of this ability. In order to do this, confirmatory factor analyses with subsequent model tests were performed. The model presented in Figure 1 seems to have the best fit for all three transformational geometry concepts. Specifically, as expected, for each geometric transformation (translation, reflection, and rotation), there are four first-order factors: 1) "recognize the image", 2) "recognize the transformation" (translation, reflection or rotation), 3) "identify the parameters", and 4) "construct the image". Two of the expected factors, "recognise the image" and "recognise the transformation" among other alternatives, seem to constitute a second order factor, which contributes significantly to ability at the primary school level. This factor was named "recognise properties" at the primary school level, since the common characteristic shared by these tasks is the recognition of each transformations' properties regarding the preservation or change of the orientation, position, and/or size of the figure. Further confirmatory factor analyses with students' mean scores for each of the twelve factors (4 factors for each geometric transformation) confirm that "translation ability", "reflection ability", and "rotation ability" all load in a higher order factor, which is considered to be "transformational geometry ability" (CFI = .96, x^2=71.90, df =52, x^2/df =1.39, RMSEA = .05). The factor loadings and their interpreted dispersion are .95 (.91) for the "translation ability" factor, .98 (.97) for the "reflection ability" factor, and .90 (.81) for the "rotation ability" factor.

The second aim of this study was to describe primary school students' levels of abilities in transformational geometry. In order to do this, RASCH dichotomous analysis was performed. The

RASCH analysis of the data suggested that it fits the model well ($\bar{X}$ =.00, SD=1.81, Infit Mean Square=.99, Outfit Mean Square=1.01, Infit t=-.06, Outfit t=.12, Reliability of Estimates=.98). Figure 2 presents the scale that resulted from this analysis.



* Factor loading values for the translation ability model (CFI = .95, x^2=136.30, *df* =98, x^2/df =1.39, RMSEA = .05)
** Factor loading values for the reflection ability model (CFI=.93, x^2= 125.34, *df*=98, x^2/df=1.28, RMSEA=.04)
*** Factor loading values for the rotation ability model (CFI=.96, x^2 = 90.62, *df*=70, x^2/df=1.30, RMSEA=.04)

Figure 1: The proposed model of ability for the three geometric transformations (translation, reflection and rotation)

On the left side of the figure, the students are ranked according to their ability. Each X represents one student. On the right side of the figure, the items of the test are ranked according to their level of difficulty. More able students, i.e. those that correctly answered more items, are at the top of the scale while less able students are at the bottom. Similarly, items that were harder for the students are at the top of the scale while easier items are at the bottom. Each item is coded as a string of three symbols. The first symbol is a letter, which indicates the type of transformation: T for translation, F for Reflection and R for rotation; the second symbol is a number, which indicates the type of task, according to the factors described in the previous section: 1 for "recognize image", 2 for "recognize transformation", 3 for "identify parameters", and 4 for "construct image". The last symbol is a number from 1 to 5, indicating the serial number of the item in the corresponding factor.

The dotted lines mark the different levels. There seem to be five levels of abilities: L1 (-5.0 to -2.5 logits), L2 (-2.49 to -0.9 logits), L3 (-0.89 to 0.89 logits), L4 (0.9 to 2.49 logits) and L5 (2.5 to 5.0 logits). After examining the assumptions suggested in literature that what forms the levels of abilities can be either the type of transformation or type of task, which did not give a clear picture of the levels, we decided to compare the levels regarding the geometric work and the visualization processes required for solving the tasks that were grouped in the same level. Hence, we studied the similarities of the tasks that were grouped in the same level and we drew on the GWS framework (Kuzniak, 2006) to understand how students could have approached the task and the idea of dimensional deconstruction (Duval, 2005) on how they visualised the objects. To name the levels, we were influenced by Edwards' (2003) terminology in the field of transformational geometry. Thus, they were named: 1) wholistic image, 2) motion of an object, 3) mapping of an object, 4) mapping of the plane, and 5) self-regulated mapping of the plane, for reasons that are explained further on.

In L1, "wholistic image", students seem to visually conceive simple relations of up-down and left-right between shapes, that exist in the real world, but without understanding neither the properties of the transformation nor of the geometrical figures. The focus is not on the shapes, but on their places as part of a whole image. They seem to visualize the plane and the objects as a whole image, as a realistic photograph in the physical world. They cannot deconstruct this figure into units, and they see the grid which represents the plane and the images as a concrete part of it, but without motion. Students at this level seem to work within Geometry I. In L2, "motion of an object", students begin to detach the shape from the plane and are able to visualize it moving on top of the plane. The emphasis is still on the shape as a tangible object, but the students can visualize the dimensional deconstruction of the representation to two separate 2D figures: the plane and the geometrical shape. Hence, we may have a 2D/2D deconstruction. However, work in this level is still within Geometry I.

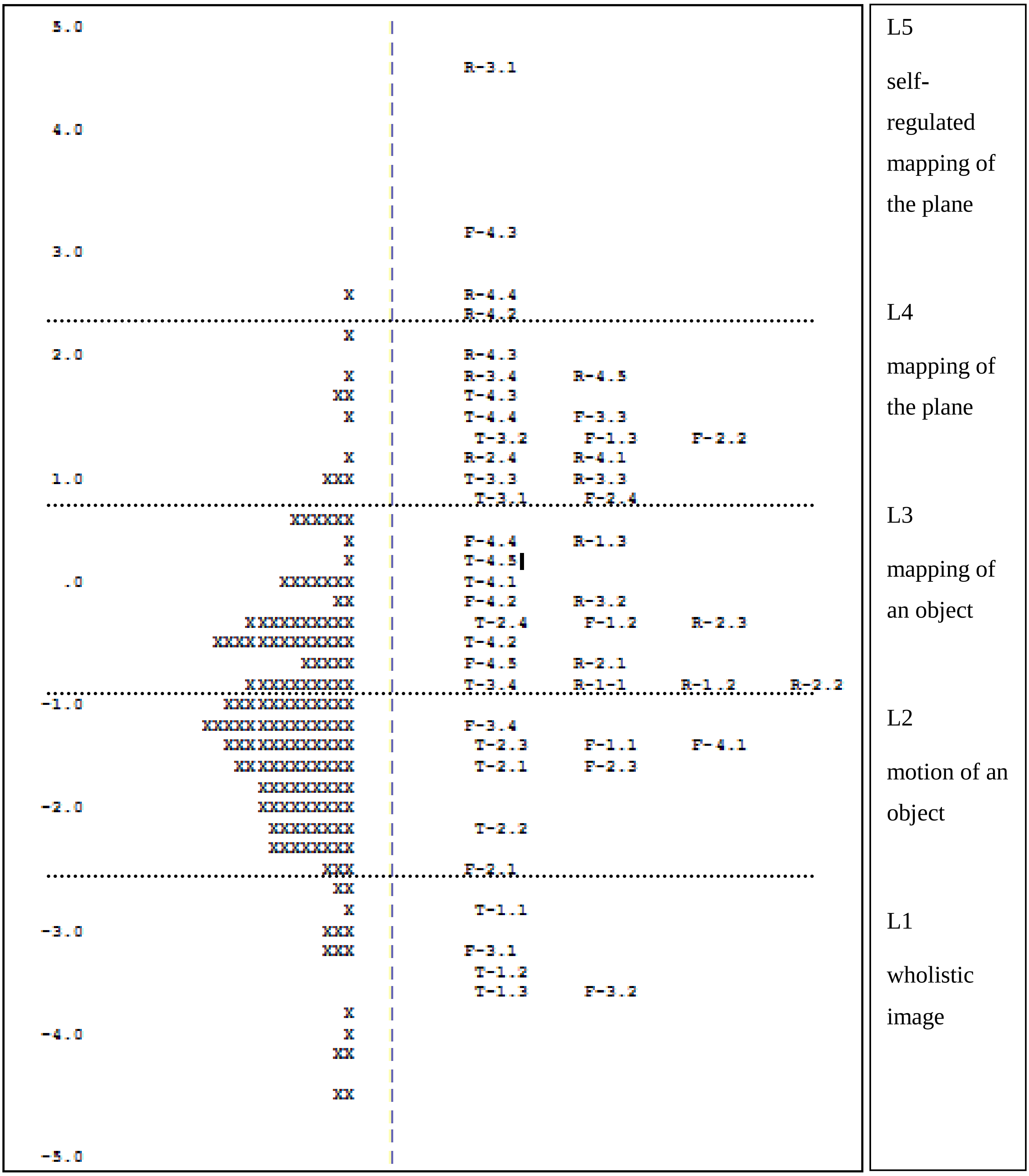


Figure 2: The scale of abilities in the transformational geometry concepts

In Level 3, “mapping of an object”, students begin to dimensionally deconstruct the 2D shape into 1D sides, and focus on the sides and their mapping. They seem to be able to map a single side and reconstruct the image of the geometrical shape based on its definition and attributes (right angles, measures etc). They begin to intuitively realize the transformation properties related to direction, orientation and distance, and they can apply them in simple situations such as constructing images in straight-line displacement and in recognizing circular displacement. Moreover, they begin to realize properties of space in the sense that shapes are made of smaller segments which can change position in space, and that the attributes of a geometrical shape are preserved in space. However, since these students have not received much formal instruction on geometric transformation properties and strategies, and their conceptions are intuitive, they are still working in Geometry I.

Students at L4, “mapping of the plane”, seem to be able to deconstruct the 2D geometrical shape into 0D points and they understand the mapping of all the points, based on the properties and axioms of transformational geometry. The shape in not anymore perceived only as a whole, but they still visualize the plane as an object. They begin to discover and apply axioms and properties to all the points of the shape, even in complex circular displacements. At this level, all students were sixth graders and they may have begun to develop some abilities and intuitive elements that could be part of Geometry II, but they are probably at a transitional stage between Geometry I and II. At Level 5, “self-regulated mapping of the plane”, students begin to have a flexible visualization of space, and they are able to dimensionally deconstruct both the geometrical shape and the plane into 0D points. They can visualize and perform the mapping of all the points in various routes and direction (straight and circular displacements), since they realize that the transformation affects all points of the plane. They seem to have some metacognitive control over their mental images of space and they can flexibly change between figural units of visualization and strategies to construct the image of a transformation. It should be noted that only one student, a sixth grader, falls within this category. Although we are not aware of many details regarding this student’s cognitive profile, it is possible that his/her cognitive abilities may enable him/her to a personal GWS that is different from other students, perhaps with characteristics closer to Geometry II.

5. Discussion

The aim of this paper was 1) to investigate the components that synthesize primary school students’ transformational geometry ability, and 2) to describe primary school students’ levels of abilities in transformational geometry. In this section, we discuss the conclusions of our findings.

Regarding the first aim, our findings confirm Kidder’s (1976) position that transformational geometry ability is multifaceted. It seems that the three geometric transformations are composed by similar factors, namely recognition of image, recognition of transformation, identification of parameters, and construction of image. Moreover, they seem to have a similar structure, with recognition of image and recognition of transformation factors forming a higher order factor – recognition of properties. This higher order factor and the two factors of identification of parameters and construction of image comprise ability in translation, reflection, and rotation respectively. Moreover, the three factors of ability in each geometric transformation load on a higher order factor, which is transformational geometry ability.

For the second aim, we adopted the theoretical framework of GWS (Kuzniak, 2006) to interpret students' levels of ability in transformational geometry concepts. Five levels of abilities were found in relation to visualization and dimensional deconstruction in geometric transformations. Our findings suggest that students at primary school level, and especially at the lower levels of abilities, seem to work within Geometry I and are strongly influenced by the natural world, even in their mental images. However, it seems that not all students that think within the same paradigm are at the same level of abilities. What seem to differentiate these levels may be some cognitive developmental abilities that form students' personal GWS, since the reference GWS does not emphasize instruction in geometric transformations and these differences cannot be attributed to teaching. Hence, even though students are not expected by the system to be working within Geometry II for the concept of geometric transformations, some students at the higher levels may possess some intuitive elements of Geometry II regarding the visualization of figures and space, or they may have connected it with knowledge from other fields of geometry. The fact that even though the reference GWS is at Geometry I and some students' personal GWS may be starting to have elements of Geometry II could suggest that sixth grade students are ready to be introduced to a more formal instruction on geometric transformations within Geometry II. This should be taken into consideration by curriculum formers in the design of geometry curricula. However, this does not mean that students at the higher levels do not approach the easier tasks within Geometry I. Further research with qualitative data of students' arguments is required to clarify primary school students GWS paradigms when solving transformational geometry tasks. Moreover, further studies of students'cognitive profile at each level may reveal reasons for the differentiation beween levels. Such studies could focus on students' spatial ability, which is considered to be highly related to transformational geometry ability (Kirby & Boulter, 1999)

Our findings are important for the teaching of transformational geometry in primary and secondary school. They support that the theory of GWS can be a useful epistemological tool in understanding students' work, abilities and difficulties in transformational geometry, and guide teachers into adjusting their teaching methods to help their students achieve higher levels of performance. Moreover, it provides evidence for the importance of practicing students' ability to identify figural units in educational frameworks for the teaching of geometry in general, which according to Duval (2011) is a fundamental principle in the learning of geometry. Hence, we should reflect about a new approach for introducing geometry in primary and secondary levels, whose principle would be that the awareness of the different ways of looking at figures is prior to the knowledge of the classical basic figures (Duval, 2011).

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