THE STRUCTURE OF ELEMENTARY STUDENTS' ABILITY IN GEOMETRIC TRANSFORMATIONS: THE CASE OF TRANSLATION

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Research in the field of geometric transformations has decreased substantially during the last years, leaving unanswered questions concerning students' ability in rigid geometric transformations. Based on the results of a large scale project investigating rigid transformations (translation, reflection, rotation) in elementary education, this paper focuses on describing the components that synthesize students' ability in translation. The sample of the study were 166 elementary school students from fourth, fifth and sixth grade, who were given a test in translation, reflection and rotation. The results suggest that geometric translation ability at elementary level is synthesized by three basic components: recognise properties, construct image, and identify parameters. The paper also discusses students' ability in each of the components.

INTRODUCTION

The growing emphasis on geometry teaching during the last few decades has modified its traditionally Euclidian-based content, by introducing new types of geometry, such as transformational geometry (Jones, 2002). According to NCTM's *Principals and Standards for School Mathematics* (2000), "Instructional programs from kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (p.41). However, there are several suggestions that there is limited research on transformational geometry (Boulter & Kirby, 1994), which is imputed to its underemphasis in mathematics curricula.

According to Kidder (1976), performing transformations is a multi-faceted mental operation. There are studies describing students' strategies in different types of rigid transformations tasks (Edwards, 1990). However, the components that synthesize this ability appear not to have been clearly defined in literature yet. This paper is based on the pilot results of a large scale project investigating the structure of students' ability in rigid transformations (translation, reflection, rotation). Its purpose is to investigate the components that synthesize elementary school students' ability in geometric transformations and the structure of this ability. The main aim of this paper is to develop and quantitatively test a theoretical model for translation ability by using a sample of students at the upper level of elementary education (fourth, fifth and sixth grade students). It also aims to describe elementary students' performance in each geometric translation ability component.

THEORETICAL FRAMEWORK

The inclusion of transformational geometry in mathematics curricula in the early 70's raised an emphasis around the importance of teaching and understanding geometric transformations (Jones, 2002). Early studies focus on providing evidence for suggesting that teaching geometric transformations in elementary and high school education is feasible and may have positive effects on students' learning of mathematics (Edwards, 1990; Williford, 1972). Later studies focus on more psychological aspects, such as students' ability and misconceptions. While the first of these studies were mostly interested in students' ability to perform transformations (Kidder, 1976; Moyer, 1978), researchers soon became more interested in examining students' abilities and strategies in a variety of transformation tasks, such as identification and execution of transformations (Hart, 1981; Edwards, 1990), as well as in examining configurations that influence students' ability in transformational geometry (Schultz & Austin, 1983). Schultz and Austin (1983) suggest that the level of difficulty for transformation tasks is influenced by the direction (vertical, horizontal, diagonal) and the size of the transformation. The latter can raise great difficulties to students, especially when the size of the transformation is so small that an object and its image are overlapping (see Figure 1 for examples in translation).

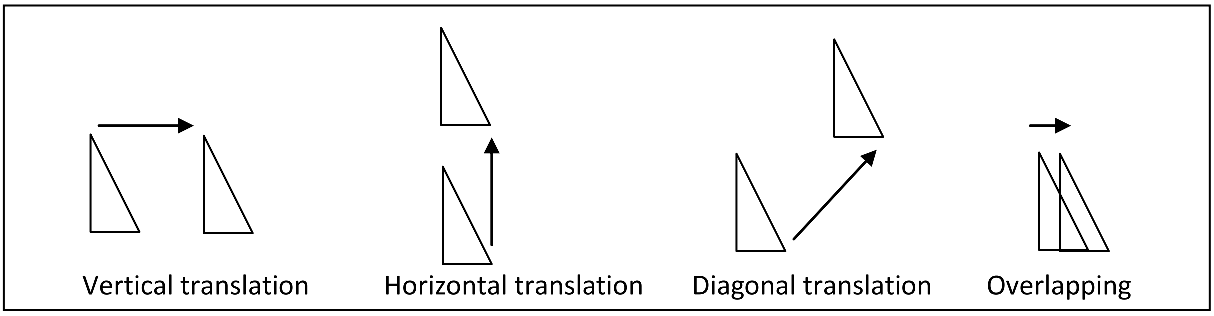


Figure 1: Configurations influencing students' ability in geometric translations

It seems that research in geometric transformations decreased substantially around the late 80's, leaving unanswered questions on the cognitive development of transformations (Boulter & Kirby, 1994). For instance, Moyer (1978) emphasized the need to search for a successful sequence of learning activities in transformational geometry for children. Some studies have attempted to describe such sequences, based on different types of tasks (Molina, 1990; Yanik & Flores, 2009). Yanik and Flores (2009), focusing on translation in particular, describe such a sequence for the knowledge and understanding of translation by a prospective teacher. However, the components that synthesize this ability have not yet been identified quantitatively. It is the aim of this paper to investigate the structure of translation ability for elementary school students, by considering 1) different types of tasks used in previous studies (Hart, 1981; Edwards, 1990), and 2) configurations found in literature (direction and size of translation) that may influence translation ability (Schultz & Austin, 1983).

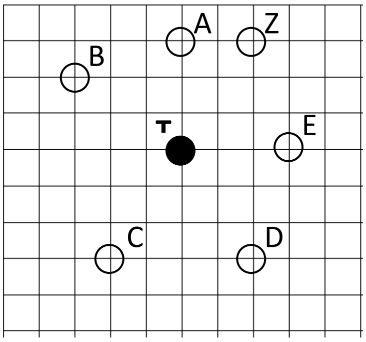
METHODOLOGY

The sample of the study was 166 elementary school students (78 boys and 88 girls). Specifically, fifty-two were fourth-graders, fifty-three were fifth-graders and sixty-one were sixth graders. All students came from urban schools in Cyprus.

The instrument used in the study was a transformational geometry test, developed especially for the purpose of the project. The test had three sections: the first section was about translation, the second about reflection and the third about rotation. Each section included four different types of tasks: 1) recognising the image of a translation/reflection/rotation among other choices, 2) recognising a translation/reflection/rotation among other choices, 3) constructing the image for a given translation/reflection/rotation, and 4) identifying the parameters (relation between image and pre-image) of a given translation/reflection/rotation (see Table 1 for examples in translation for each category). For each type, at least three tasks were given: one in horizontal, one in vertical and one in diagonal direction. In types 3 and 4, there was an additional task with overlapping image and in type 3 an additional task with an unfamiliar shape in horizontal direction. The tasks were split and administered to all students in two equally difficult parts, approximately one week apart. Students were given forty minutes for each part of the test. In order to avoid practice effects, half of the students received one part first, while the other half received the other part.

After completing the test, students' responses were graded. Since this paper focuses on geometric translation, only the grading procedure for the translation tasks will be presented. Types of tasks 1) recognising the image of a translation (3 tasks), and 2) recognising a translation (4 tasks) were multiple choice tasks, with four alternative responses. In these tasks, 0 marks were given to each incorrect response and 1 mark to each correct response. In type 3, constructing the image for a translation (horizontal, vertical, overlap and unfamiliar shape items), grading was: 0 marks for incorrect or no response, 0.2 for correct image with false orientation, 0.4 for correct image in false direction, 0.6 for correct image in false distance, 0.8 for correct image with inaccuracies in the size of the shape, 1 for correct response in all the aforementioned parameters. In the case of the Type 3 task with diagonal direction, the grading was: 0 marks for incorrect response, 0.25 for correct image with false direction in one dimension (either up-down or left-right), 0.50 for correct image with false distance in one direction, 0.75 for correct image with inaccuracies in the size of the shape and 1 for correct response in all the aforementioned parameters. Finally, type 4 tasks, identifying the parameters for a given translation, were graded as follows: 0 marks for incorrect response, 0.33 for finding the correct direction but false distance, 0.66 for finding the correct distance but false direction and 1 for correct response. In the case of diagonal direction, again the coding was different: 0 marks for incorrect response, 0.25 for finding the correct direction in one dimension (either up-down or left-right), 0.50 for finding the correct distance in one direction, 0.75 for finding the correct direction and distance for one dimension only and 1 for correct direction and distance in both dimensions.

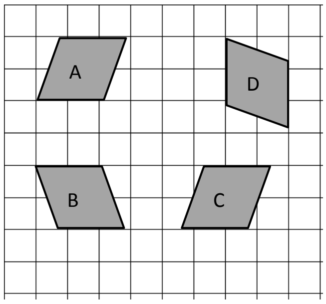
Type 1 tasks



Which of the following alternatives is the image of T when translated 3 units up and 2 units to the right? Circle the correct answer.

1. A 2. B 3. D 4. Z

Type 2 tasks

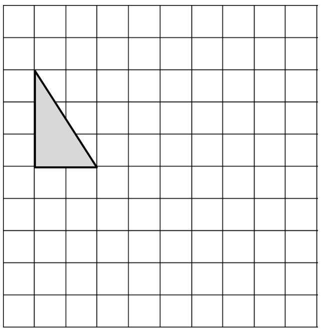


Which of the following pairs is an example of translation? Circle the correct answer.

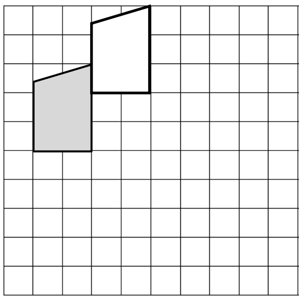
1. A and D 2. B and C
3. C and D 4. A and C

Type 3 tasks

Find the image of the shape when translated 6 units to the right.



Type 4 tasks



Give directions for translating the shaded figure to the position of the white figure.

Table 1: Types of geometric translation tasks

For testing the fit of the proposed model, the MPLUS software was used with ML estimator. More than one fit indices were used to evaluate the extent to which the data fit the theoretical model under investigation. More specific, the fit indices and their optimal values were: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, since a significant chi-square indicates lack of satisfactory model fit, (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2004).

RESULTS

The main focus of this paper is to describe and empirically test a theoretical model of translation ability by using a sample of fourth, fifth and sixth grade elementary school students. After subsequent model tests, the model shown in

Figure 2 proved to have very good fit to the data (x^2=136.303, df =98, x^2/df =1.39, CFI = 0.948, and RMSEA = 0.049).

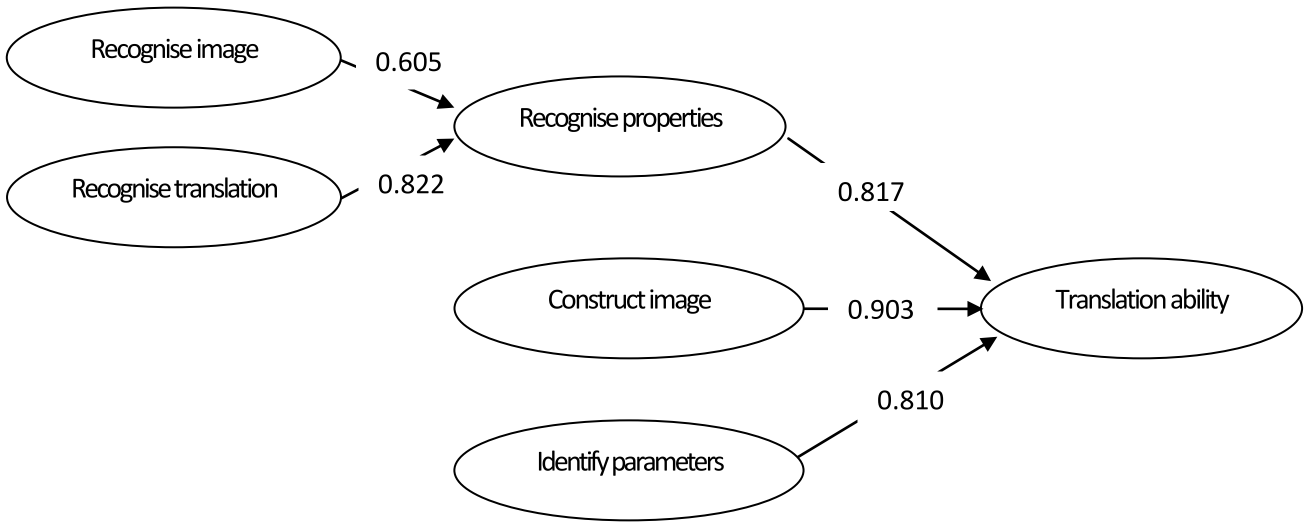


Figure 2: The proposed model of translation ability

Figure 2 presents the suggested model of translation ability. Two of the expected factors, recognise the image of a given translation and recognise a translation among other alternatives, seem to constitute a second order factor, which contributes significantly to translation ability at the elementary level. We called this factor “recognise properties” at the elementary level, since we believe that the common characteristic shared by these tasks is the recognition of the translation properties to preserve both the orientation and size of the figure. Figure 2 also shows that “construct image” appears to be a very important factor of translation ability, with the highest contribution of all (0.903). The last factor presented in Figure 2, which also contributes to translation ability, is named “identify parameters”. The coefficients could serve as an indicator of the importance of each factor for understanding translation. All coefficients are high (loadings are greater than 0.7) showing that all factors have a considerable contribution towards translation ability. Moreover, all coefficients are statistically significant at level 0.05.

Table 2 presents the fit-indices of the one factor model and the proposed model. The one factor model assumes that all variables load on a single first order factor, which would be geometric translation ability, and that there are no sub-factors comprising this general ability. This would mean that translation ability is uni-dimensional. The comparison of the two models, suggests that the proposed model fits to the data much better than the one factor model, since the one factor model fit indices are not within acceptable values (x^2/df < 1.96, CFI > 0.90, RMSEA < 0.06). This supports Kidder’s (1976) suggestion of transformations being a multi-faceted construct. Moreover, in this case, the results suggest that even a single type of transformations, namely translation, has its own structure of different components.

	x^2	df	x^2/df	x^2 difference	df difference	CFI	RMSEA
One factor model	379.127	104	3.65			0.628	0.126
Theoretical model	136.303	98	1.39	242.824	6	0.948	0.049

Table 2: Alternative models of translation ability

Table 3 presents the means of performance for students in each component. What becomes apparent from the table is that the tasks of recognising the image seem to be the easiest for students ($\overline{X}$ = 0.855, SD = 0.311), and that the most difficult tasks appear to be constructing the image for a given translation ($\overline{X}$ = 0.470, SD = 0.281). Some of the students' most common mistakes in the case of recognising the image of a given translation were to translate the shape to the wrong direction, or in the case of a diagonal displacement, only one direction. In the case of recognising translation, the most common mistake was to confuse it with reflection. In the case of constructing the image, the most common mistake was to start counting from the point on the right of the shape, to count the correct number of units and then start drawing the image from the point which was on the left of the shape. Hence, although the image was correct regarding size and orientation, and the direction of the translation was correct, this strategy resulted to a false distance measure. In identifying parameters, the most common mistake was to count only the distance units that were between the shape and its image to find the size of the translation. This was more obvious in the case of overlapping figures, where students often commented that "the shape did not translate at all" or that "it was translated on top of the original shape".

Component	$\overline{X}$	SD
Recognise image	0.855	0.311
Recognise translation	0.564	0.337
Construct image	0.470	0.281
Identify parameters	0.500	0.246

Table 3: Means and standard deviations of performance for components

DISCUSSION

The aim of this paper was develop to a model of geometric translation ability at elementary level education. The findings of this research suggest that elementary school students' geometric translation ability can be described by three main components, namely recognise the properties of translation, construct the image of a figure and identify the parameters of a given translation. The structure of this model supports Kidder's (1976) position that performing transformations is a multi-faceted mental operation, in this case in the context of translation.

The first component is recognising the properties of translation. It consists of two sub-components: 1) recognising the image of a translated figure and 2) recognising a translation. The common features underlying these two sub-components, and consequently the component of recognising the properties of translation, are students' identification of the preserving characteristics of the shape when it is displaced. These are the preservation of the orientation of the figure in space and the preservation of its size (Schultz & Austin, 1983). In both sub-components' tasks, the students need to compare the original figure to its image and decide whether these characteristics remain the same. This component of translation ability and its sub-components seem to be the easiest for students to perform.

The second component of translation ability is constructing the image of a given translation. It relates to students' ability to perform a translation, by considering all properties and configurations simultaneously. This means that students need to execute the translation by following the correct direction and counting the correct number of units, while at the same time preserving the shape, orientation and size of the figure. This complexity is probably what makes the tasks of this component the most difficult for the students, since at this point they appear to have the lowest mean performance.

The third component is identifying the parameters of a given translation. In this type of tasks, the students take the properties (orientation and size of figure) for granted, since they already know it is a translation and both the figure and its image are given. The focus of their attention here shifts to defining the two parameters of translation, which are the direction and the size of the translation. Therefore, they need to realize the correct direction and then count the distance units between at least two corresponding points.

The findings of the present study are important both for teaching as well as for assessing geometric translation ability. Teachers should consider all components synthesizing this ability when designing their instruction. Further research could focus on finding effective ways of teaching for promoting all components of translation ability, as well as an effective sequence for teaching all components at elementary education.

ACKNOWLEDGMENTS

This work falls under the Cyprus Research Promotion Foundation's Framework Programme for Research, Technological Development and Innovation 2009 - 2010 (DESMI 2009-2010), co-funded by the Republic of Cyprus and the European Regional Development Fund (Grant:PENEK/0609/57).

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